FUNCTORIAL RELATIONSHIPS IN THE ANATOMY OF COGNITIVE AND BEHAVIORAL PSYCHOLOGY

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ABSTRACT
The functorial relationship between real-world environments and our cognitive mind reveal some powerful mathematical structures and functions inherent in the mind that facilitate psycho-behavioral dynamics. In this work we explore some theorems found in Category theory and their applications to cognitive psychology. Through a likely scenario of a human experience with typical “objects” and human-choice considerations, we apply constructions found in the Yoneda lemma and Representability theorems, in the context of cognitive psychological sciences. We also explicate the role that these “Yoneda” constructions play in the psychobehavioral dynamics of business marketing and customer relations. In the spirit of mathematical psychology, we clearly state which objects reside in the physical space and those that “reside” in the cognitive mental space.

Key Words: *Mathematical Psychology, Category Theory, Representable Functor, Yoneda lemma, Cognitive Psychology, Behavioral Psychology, Consumer Psychology.*

0. INTRODUCTION
Psychologists agree that the worlds of physical phenomena and mental phenomena are two different classes of being (Lundh, 2018, p. 52) (Eysenck & Keane, 2000, p. 266). Abstract results in category theory such as the lemma due to Nabuo Yoneda, the Yoneda lemma, provide constructions that are suitable to mathematically study cognitive psychological and psychobehavioral dynamics, and relations in and between the mental space and the physical real-world environments. For instance, we like to think that “proper” reasoning and action about and in the world around us, and indeed survival in our physical environments, depends on whether or not our perception “properly” represents (presents) our environment to us in the form of mental representations. Can we mathematically show that our perception “properly” represents to us the real-world entities and their relationships? We explore some constructions found in the Yoneda lemma, and theorems in Category theory that describe a criterion to determine if a given person’s perception can be said to be “Representable” or not, defined as a functor from a categorical real-world environment into a categorical cognitive mental space. We also explicate the role that these “Yoneda” constructions play in the psychobehavioral dynamics of business marketing and customer relations.

Mathematically, a category involves a collection of objects and their relations, where the “elements” of those objects, if they have any at all, are typically not necessary to identify to work within or across categories. However, Yoneda’s lemma, and constructions supplied in the spirit of that lemma provide excellent tools to study the relationship between the categorical point of view and an element-wise point of view (Drossos, 1987, p. 107). The Yoneda lemma is such a general result in category theory that it exists in and unifies various areas of mathematics, the sciences, and indeed various philosophies (McLarty, 2006, p. 25)(Songa, 2012). Such broadness of the lemma is expressed in the following research studies.

A psychosocial study in Interdependence theory, by Rusbult and Lange (2003, p. 351), involved situation-based interaction between partners A and B, examining the effects and interaction of each person's behavior to determine the impact on their own self, on each other, and the impact of their joint-action on one or the other person. Rusbult and Lange define four properties of the situation structure based on dependence relations between partners A and B. They found that each partners' situation-based behavioral pattern, along their relational dependencies, logically implies the existence of personal traits, relevant goals, motives, and abilities of each partner. These situations are instances of Yoneda constructions for partners A and B; where, the situation is a physical category, A and B are objects of the situation and their behaviors can be functorial, denoted F(A) and F(B) respectively. According to the Yoneda lemma, their personal "traits" T(A) and T(B) can be determined in a one-to-one fashion with respect to a set of dependence relations between A and B. For example, during situations involving joint-control dependency, some personal traits of B can be "critical thinker", "problem solver", and "takes initiative", determined by how B behaves in coordination with A's actions with respect to some joint dependency rules.

In (Drossos, 1987, p. 107), the dynamics between physical real-world categories (holistic-geometric) and mental categories (logical analytic) has been studied. The natural transition from a holistic (geometric) consideration of objects and their relationships in their "big picture" structure, to the element-wise (logical-analytic) consideration of aspects (elements) of those objects, is a typical human psychological "functorial" process between two related categories of interest to humans.

A category C is only a category, by human psychological consideration of its objects and relations. According to Piaget, the geometric "holistic" category is a more primary determination in the early course of human mental development, than the logical analytic category that is based on language and propositional logic (Piaget, 1975, p. 213). Lawvere admits that while the formation of the Holistic (Geometric) categorical constructions are anterior to the Logical type categories, the Logical categories are a special case of the Geometric (Lawvere, 1970, p. 329). In our work, we will require that all persons (individuals) are at an intellectual stage where they exercise their ability to "create", perceive, or determine both Geometric-type and Logical-type categories.

In 2013 Andreatta, et. al., established a category-theoretic model that covers three domains- musical creativity, discourse theory, and cognition. Their application of category theory to the creative work of Beethoven (piano sonata opus 109) showed that the categorical object called the “colimit” is the unifying universal construction in the three domains, and “the central role played by the Yoneda lemma” (Andreatta et al., 2013, p. 19). In the case of musical creativity, Yoneda’s lemma applies when representable presheaves are restricted to small fully faithful subcategories. In discourse theory, the production of “sense” depends on generalizing from Yoneda constructions of presheaves as colimits of representable presheaves to shape colimits. In the theory of cognitive neuroscience, hierarchal systems of neural networks generate new neural objects called “category-neurons”, as colimits of lower level neural networks. The categorical model of Adreatta, et al, suggests a platform for experimentation to empirically verify the theoretical result of the colimit construction over the three categorical domains.

(Seremeti & Kougias, 2013) applied the Yoneda philosophy to mathematically study a criterion for building ontologies that are consistent and accurate in providing the conceptual frame work for representing specified domains of interest. That is, what are the qualifications for an entity to be an object in an ontology (category)? In their work an ontology is a category, where its objects are concepts (classes of entities) that have functional relationships between those classes, in a hierarchal lattice structure. The formal determination of a concept as a building-block object in an ontology (category) is guided by the link between the Yoneda Embedding lemma and Formal Concept Analysis. In the work of Seremeti and Kougias, attributes of a thing are assumed to be concepts- “units of thought gained by abstraction” (Cimiano et al., 2004, p. 189). A binary relation between classes (concepts) is represented by a formal context. A formal context, in the theory of Formal Concept Analysis, can be a triplet $< X, Y, R >$, where X is the set of entities that belong to a concept C, Y is the set of all attributes shared by all entities in X, and R is a binary relation such that for each $w \in X$ and $k \in Y$ the pair $(w, k) \in R$ if and only if “ w has attribute k”.

With the attributes, the entities, the concepts, and the relation all being units of thought in the ontology (category), where attributes k and entities w are considered to be objects, results from Yoneda constructs apply that qualify or disqualify an entity’s “object” status in the ontology. For instance, for any $(w, k) \in R$ and $(p, k) \in R$, then there are incoming binary relations (arrows) from the concept objects w and p to the attribute object k, written $w \to k$ and $p \to k$. A result inferred from the Yoneda Embedding lemma is that an object of a category is strictly and completely determined by its incoming arrows (Seremeti & Kougias, 2013).

In many human experiences one must act in/on an existing physical category (domain of interest) with a specified objective or goal to achieve; for example, in a collection of institutions/companies and their relations with one another, that a person must act within in order to achieve and maintain a car of desired quality. One would have to construct a mental ontology, of units of thought, to properly conceptualize (represent) the existing physical category in order to achieve goals in that physical domain of interest, which infers the necessity of representable functors (sheaves) between the physical domain of interest and the mental ontology (Andreatta et al., 1996, p. 19). The conscious or "unconscious" choice of objects to include in one's mental ontology is a result supported by Yoneda constructs (Seremeti & Kougias, 2013).

In the following Section 1.1 we present a scenario where a mental category is constructed from a possible physical real-world environment and we analyze some functorial properties of perception with respect to Yoneda constructions. In Section 1.3 we define a functorial adjoint to the perception functor, in the context of behavioral psychology. In Section 2 we analyze psychobehavioral applications in business that detail the practical use of mapping objects to sets of relations- a common construction at the heart of Yoneda's lemma.

Before we move on to our scenarios, we make a note on the physical environmental spaces and human cognitive mental spaces that are under consideration, here. The physical environmental spaces and human cognitive mental spaces are considerably much more richer than spaces modeled by the typical set theory. To enjoy the generality of such spaces, our physical universe denoted $\mathrm{R}^3(\mathrm{t})$, and our space of mental representations, denoted M- our "mind", are considered in this work to be Grothendieck Universes (Williams, 1969, p. 1). That is,

1. If $\mathrm{A} \in \mathrm{R}^3(\mathrm{t})$ and $\mathrm{x} \in \mathrm{A}$, then $\mathrm{x} \in \mathrm{R}^3(\mathrm{t})$,

2. If $\mathrm{A}, \mathrm{B} \in \mathrm{R}^3(\mathrm{t})$ then $\{\mathrm{A}, \mathrm{B}\} \in \mathrm{R}^3(\mathrm{t})$,

3. If $\mathrm{A} \in \mathrm{R}^3(\mathrm{t})$, then the power set of A , $\mathcal{P}(\mathrm{A}) \in \mathrm{R}^3(\mathrm{t})$, and

4. If $\mathrm{f}: \mathrm{X} \rightarrow \mathrm{Y}$ is any relation between objects $\mathrm{X}, \mathrm{Y} \in \mathrm{R}^3(\mathrm{t})$, then that relation $\mathrm{f} \in \mathrm{R}^3(\mathrm{t})$.

Rules 1 – 4 also hold in the mind M.

1. CATEGORICAL CONSTRUCTION AND PERCEPTION REPRESENTABILITY

Our first categorical construction and exploration begins with a likely scenario of a human experience with a simplified category of typical objects and relations that must be acted on in order to achieve and maintain a car of desired quality.

The construction is supported by a brief but definitive statement by Augustus De Morgan in 1853:

> "When two objects, qualities, classes, or attributes, viewed together by the mind, are seen under some connexion, that connexion is called a relation" (Heath, 1966, p119).

1.1 Scenario 1.

Let $R^3(t)$ be the 3-dimensional space-time "real-world". Let $\mathbf{E}$ be any finite perceivable environment in $R^3(t)$, written $\mathbf{E} \subset R^3(t)$. Say $\mathbf{E}$ is perceivable to person A, let the following entities be "objects" in $\mathbf{E}$ that A knows (perceives) to be related to A's car;

O_1 = person A's car; O_2 = A's bank; O_3 = credit bureaus; O_4 = A's revenue source; O_5 = A's education source; O_6 = car service; O_7 = car fuel; and O_8 = A's creditors.

For each object O_i, let the set $\{a_{i_k} \mid k = 1, 2, 3, \ldots, d\}$ be the set of physically "perceivable" aspects of O_i that "can be" perceived by person A.

Postulate: For every $O_i \in \mathbf{E}$ let there be an arrow between O_i and O_1 if and only if there is some perceived aspect a_{i_k} of O_i that is related to some perceived aspect a_{1_k} of O_1, written $O_i \to O_1$.

Similarly, let $O_n \to O_m$ iff there is some perceived aspect a_{n_k} of O_n that is related to some perceived aspect a_{m_k} of O_m.

A human mental activity that we mention here is the "art of postulating". In a more general use of the word, we consider a postulate to be any mental or physical structure construed by the human mind, then taken to be true, that can or must be acted in/on to achieve a desired outcome. In this sense the word postulate is a gerund- a structure construed by the human mind, and the mental act of creating such a structure.

Proposition 1. The class of objects $O_i \in \mathbf{E}$ and their perceived relations (arrows) is a category, see Figure 1.

Definition 1 (Perception). For any object $O_i \in \mathbf{E}$ define perception P to be a map from the environment $\mathbf{E}$ to a cognitive mental subspace $\mathbf{C} \subset M$, where M is person A's mind, written $P: \mathbf{E} \to \mathbf{C}$. For any $O_i \in \mathbf{E}$, define $P: \mathbf{E} \to \mathbf{C}$ to be

$$P(O_i) = \{q_{i_k} \mid q_{i_k} \text{ is a mental representation of a perceivable aspect } a_{i_k} \text{ of } O_i\} \in \mathbf{C},$$

for $k = 1,2,3 \ldots h_i$; and, for any two objects $P(O_n), P(O_m) \in \mathbf{C}$, if g is an arrow in $\mathbf{E}$ defined by $O_n \xrightarrow{g} O_m$, then let Pg be the arrow in $\mathbf{C}$ defined by

$$P(O_n) \xrightarrow{Pg} P(O_m),$$

where $Pg(q_{n_k}) = q_{m_k}$ for some $q_{n_k} \in P(O_n)$ and $q_{m_k} \in P(O_m)$. That is, Pg is the mental image of the perceived relationship between aspects of O_n and O_m.

As a map, $\mathrm{P}: \mathbf{E} \rightarrow \mathbf{C}$, P explicitly maps physical perceivable aspects $\mathrm{a}_{\mathrm{i}_\mathrm{k}} \in \mathrm{O}_\mathrm{i} \in \mathbf{E}$ to a mental representation $\mathrm{P}(\mathrm{a}_{\mathrm{i}_\mathrm{k}}) = \mathrm{q}_{\mathrm{i}_\mathrm{k}} \in \mathrm{P}(\mathrm{O}_\mathrm{i}) \in \mathbf{C}$, in the mind of some person A perceiving O_i (Eysenck & Keane, 2000, p. 266).

In many mathematical constructions a sheave $\mathrm{F}: \mathrm{X} \rightarrow \mathrm{Y}$, on a topological space X, maps open sets $\mathrm{V} \subseteq \mathrm{X}$ to an Abelian group $\mathrm{F}(\mathrm{V}) \subseteq \mathrm{Y}$, satisfying certain restriction map properties on subsets of V. With respect to (Sims et al., 2018, 2019), our perception $\mathrm{P}: \mathbf{E} \rightarrow \mathbf{C}$ can be a sheave on an R_0-topological environment $\mathbf{E}$, that maps clopen objects $\mathrm{O}_\mathrm{i} \in \mathbf{E}$ to generalized mental Algebras $\mathrm{P}(\mathrm{O}_\mathrm{i}) \in \mathbf{C}$, satisfying certain "mental grammar" rules on mental representations $\mathrm{q}_{\mathrm{i}_\mathrm{k}} \in \mathbf{C}$ and formulas $\phi(\mathrm{q}_{\mathrm{i}_1}, \mathrm{q}_{\mathrm{i}_2}, \ldots \mathrm{q}_{\mathrm{i}_\mathrm{z}}) \in \mathbf{C}$ of mental representations.

Proposition 2. The class of objects $\mathrm{P}(\mathrm{O}_\mathrm{i}) \in \mathbf{C}$ and their perceived relations (arrows) is a category.

Proposition 3. Perception P is a functor from categories in the real-world environment $\mathbf{E} \subset \mathrm{R}^3(\mathrm{t})$ to categories in the cognitive mental subspace $\mathbf{C} \subset \mathrm{M}$.

Definition 2 (Universal Element). Given the categories $\mathbf{E}$ and $\mathbf{C}$, a universal element of a functor $\mathrm{P}: \mathbf{E} \rightarrow \mathbf{C}$ is a pair (O, q), where O is an object of $\mathbf{E}$ and $\mathrm{q} \in \mathrm{P}(\mathrm{O})$, such that for every pair $(\mathrm{O}_\mathrm{j}, \mathrm{q}_{\mathrm{j}_\mathrm{k}})$ with $\mathrm{q}_{\mathrm{j}_\mathrm{k}} \in \mathrm{P}(\mathrm{O}_\mathrm{j})$, there exists a unique function $\mathrm{f}_{\mathrm{j}_\mathrm{k}}: \mathrm{O} \rightarrow \mathrm{O}_\mathrm{j}$ in $\mathbf{E}$ such that $\mathrm{Pf}_{\mathrm{j}_\mathrm{k}}(\mathrm{q}) = \mathrm{q}_{\mathrm{j}_\mathrm{k}}$ in $\mathbf{C}$ (Mac Lane, 1998a, p. 55).

Psychologically, the unique function $\mathrm{f}_{\mathrm{j}_\mathrm{k}}$ is a relation born by a person considering, in their mind, an important connection between the pair of objects O and O_j. For that individual, $\mathrm{f}_{\mathrm{j}_\mathrm{k}}$ is a relation in $\mathbf{E}$ that importantly (uniquely) relates the perceived aspect $\mathrm{a} \in \mathrm{O}$ with aspect $\mathrm{a}_{\mathrm{j}_\mathrm{k}} \in \mathrm{O}_\mathrm{j}$, in the form of respective mental aspect representations $\mathrm{q} \in \mathrm{P}(\mathrm{O})$ and $\mathrm{q}_{\mathrm{j}_\mathrm{k}} \in \mathrm{P}(\mathrm{O}_\mathrm{j})$, via their perception of the relation $\mathrm{Pf}_{\mathrm{j}_\mathrm{k}}$.

The perceived aspects $\mathrm{a} \in \mathrm{O}$ and $\mathrm{a}_{\mathrm{j}_\mathrm{k}} \in \mathrm{O}_\mathrm{j}$, and mental representations $\mathrm{q} \in \mathrm{P}(\mathrm{O})$ and $\mathrm{q}_{\mathrm{j}_\mathrm{k}} \in \mathrm{P}(\mathrm{O}_\mathrm{j})$, and unique function $\mathrm{f}_{\mathrm{j}_\mathrm{k}}$, are personal and psychological, and are mental constructs according to personal perception and choice. In fact, the object, such as a person's car, and its perceived aspects only becomes a "universal element" because that particular individual thinks of, or is aware of, other entities (objects) whose aspects are importantly related to their car (Seremeti & Kougias, 2013).

In the theory of pointed sets, any two selected aspects $\mathrm{q}_{\mathrm{j}_\mathrm{k}} \in \mathrm{P}(\mathrm{O}_\mathrm{j})$ and $\mathrm{q}_{\mathrm{n}_\mathrm{k}} \in \mathrm{P}(\mathrm{O}_\mathrm{n})$ under consideration here, are generalized chosen base points for the pointed sets $(\mathrm{P}(\mathrm{O}_\mathrm{j}), \mathrm{q}_{\mathrm{j}_\mathrm{k}})$ and $(\mathrm{P}(\mathrm{O}_\mathrm{n}), \mathrm{q}_{\mathrm{n}_\mathrm{k}})$. The relations $\mathrm{Pg}: \mathrm{P}(\mathrm{O}_\mathrm{j}) \rightarrow \mathrm{P}(\mathrm{O}_\mathrm{n})$ are point-preserving relations where $\mathrm{P}g(q_{j_k}) = q_{n_k}$. After all, the base point of a set is selected by the observer, the person.

In Scenario 1, where O_1 is person A's car, suppose a_{1_0} is the physical aspect "quality of the car" in $\boldsymbol{E}$, then mental representation $q_{1_0} \in P(O_1) \in \boldsymbol{C}$, in person A's cognitive mind $\boldsymbol{C}$. If the other objects $O_j \in \boldsymbol{E}$ are perceived by A to be importantly related to O_1 by direct or composite relation, where there is only one relation $h_{j_k}: O_j \rightarrow O_1$ in $\boldsymbol{E}$ such that $Ph_{j_k}(q_{j_k}) = q_{1_0}$ in $\boldsymbol{C}$, for some mental representation $q_{j_k} \in P(O_j)$ that is importantly related to "car quality" representation q_{1_0}, then the pair (O_1, q_{1_0}) is a universal element- in our case a categorical terminal element.

We now analyze Scenario 1 at both the categorical and elemental levels. For instance, where $O_2 =$ A's bank, let $q_{2_0} \in P(O_2)$ be mental representation of "the amount of car loan" ;

where $O_3 =$ A's credit bureaus, let $q_{3_0} \in P(O_3)$ be mental representation of "the credit score" ;

where $O_4 =$ A's revenue source, let $q_{4_0} \in P(O_4)$ be mental representation of "salary range or cash-flow" ;

where $O_5 =$ A's education source, let $q_{5_0} \in P(O_5)$ be mental representation of "academic preparation and creative ambition" ;

where $O_6 =$ car service, let $q_{6_0} \in P(O_6)$ be mental representation of "service expertise level" ;

where $O_7 =$ car fuel, let $q_{7_0} \in P(O_7)$ be mental representation of "octane level"; and

where $O_8 =$ A's creditors, let $q_{8_0} \in P(O_8)$ be mental representation of "credit status".

Now,
A's academic preparation and creative ambition level $a_{5_0} \in O_5$ can determine A's cash-flow or salary range $a_{4_0} \in O_4$ so that there is a unique arrow $t: O_5 \rightarrow O_4$ such that $Pt(q_{5_0}) = q_{4_0}$,

A's cash-flow or salary range $a_{4_0} \in O_4$ can determine timely payments to A's creditors and credit status $a_{8_0} \in O_8$ so that there is a unique arrow $s: O_4 \rightarrow O_8$ such that $Ps(q_{4_0}) = q_{8_0}$,

A's credit status $a_{8_0} \in O_4$ with creditors can determine A's credit score $a_{3_0} \in O_3$ so that there is a unique arrow $v: O_8 \rightarrow O_3$ such that $Pv(q_{8_0}) = q_{3_0}$,

A's cash-flow or salary range $a_{4_0} \in O_4$ and A's credit score $a_{3_0} \in O_3$ can determine the amount of A's car loan $a_{2_0} \in O_2$ so that there are unique arrows $w: O_4 \rightarrow O_2$ and $u: O_3 \rightarrow O_2$ such that $Pw(q_{4_0}) = q_{2_0}$ and $Pu(q_{3_0}) = q_{2_0}$.

The amount of A's car loan $a_{2_0} \in O_2$ can determine A's car quality level $a_{1_0} \in O_1$, so that there is a unique arrow $f_{2_0}: O_2 \rightarrow O_1$ such that $Pf_{2_0}(q_{2_0}) = q_{1_0}$, and the service expertise level $a_{6_0} \in O_6$ and fuel octane level $a_{7_0} \in O_7$ can each determine the maintainence of A's car quality level $a_{1_0} \in O_1$, so that there are unique arrows $f_{6_0}: O_6 \to O_1$ and $f_{7_0}: O_7 \to O_1$ such that $Pf_{6_0}(q_{6_0}) = q_{1_0}$ and $Pf_{7_0}(q_{7_0}) = q_{1_0}$.

While the objects and aspect-relations, here, are detailed, there can be many more details that psychologically take place in a person's mind with respect to obtaining and maintaining a car of specified quality.

The following diagram, Figure 1, illustrates the functorial perception relationship $P: \mathbf{E} \to \boldsymbol{C}$, in Scenario 1. The physical car O_1 and its mental representation $\mathrm{P}(\mathrm{O}_1)$ are terminal (universal) objects in their respective categories, and the mental element $\mathrm{q}_{1_0} \in \mathrm{P}(\mathrm{O}_1)$ is a mental representation of the "car quality". The other O_n are entities "related" to the car O_1 indicated by direct or composite arrows in $\mathbf{E}$. The mental representations $\mathrm{q}_{\mathrm{n}_0}$ of their respective aspect of O_n are base points of their respective mental pointed sets $(\mathrm{P}(\mathrm{O}_\mathrm{n}), \mathrm{q}_{\mathrm{n}_0})$, identified in category $\mathbf{C}$. The pair $(O_1, q_{1_0}) \equiv (car, quality\ rep)$, not noted in the diagram, is a categorical terminal element and is therefore a universal element of the perception functor P, Definition 2.

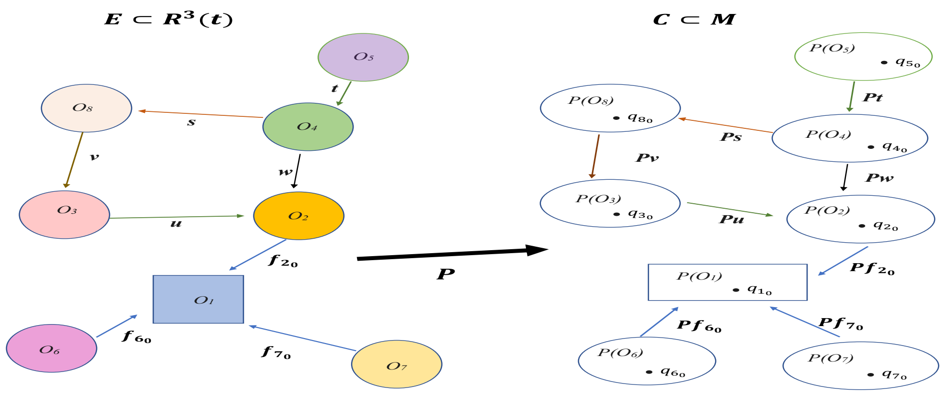

Figure 1. *The diagram illustrates the functorial perception relationship $P: \boldsymbol{E} \to \boldsymbol{C}$, from a physical environmental category $\boldsymbol{E}$ of physical entities O_n and relations, to a cognitive category $\boldsymbol{C}$ of mental entities $P(O_n)$ and relations. The physical car O_1 and its mental representation $P(O_1)$ are terminal (universal) objects in their respective categories.*

1.2 Enriched Categories

Our Grothendieck universes $\mathbf{E}$ and $\mathbf{C}$ are actually enriched categories in the sense that for any two objects O_r , $\mathrm{O}_\mathrm{m} \in \mathbf{E}$, and any relations $\mathrm{f}_{\mathrm{r}_\mathrm{s}}: \mathrm{O}_\mathrm{m} \to \mathrm{O}_\mathrm{r}$, then $\mathrm{f}_{\mathrm{r}_\mathrm{s}} \in \mathbf{E}$. That is the set of relations from O_m to O_r , $\mathrm{Hom}(\mathrm{O}_\mathrm{m}, \mathrm{O}_\mathrm{r}) = \{\mathrm{f}_{\mathrm{r}_\mathrm{s}} | \mathrm{f}_{\mathrm{r}_\mathrm{s}}: \mathrm{O}_\mathrm{m} \to \mathrm{O}_\mathrm{r}\}$, is also in $\mathbf{E}$, and $\mathbf{E}$ is closed under composition of functions. The same holds for objects and arrows in $\mathbf{C}$. Observe that the "Grothendieck" categories $\mathbf{E}$ and $\mathbf{C}$ both act like the category **SET**, the category of sets. Further, since human perception only processes finitely many "things" in finitely many steps, all collections of morphisms (arrows, relations, etc) from categories $\mathbf{E}$ or $\mathbf{C}$ are taken to be finite sets. So for perceptual purposes, $\mathbf{E}$ and $\mathbf{C}$ are also said to be "small" categories.

Definition 3 (Hom-functor). For a fixed object $O_r \in \mathbf{E}$, we may have the functorial relation $\psi: \mathbf{E} \rightarrow \mathbf{E}$ that map objects and relations from $\mathbf{E}$ into $\mathbf{E}$ defined by

$\psi(O_m) = \mathrm{Hom}(O_m, O_r)$, for objects $O_m \in \mathbf{E}$ and $\psi(f_{r_s}) = f_{r_s} \circ h$, for all relations

$f_{r_s} \in \mathrm{Hom}(O_m, O_r)$, when ever the composite $O_j \overrightarrow{\ h\ } O_m \overrightarrow{\ f_{r_s}\ } O_r$ exists.

ψ so defined is with respect to O_r, denoted by $\psi_{O_r}(\underline{\quad}) = \mathrm{Hom}(\underline{\quad}, O_r)$, and is typically called the Hom-functor.

Definition 4 (Representability). A functor $F: \mathbf{X} \rightarrow \mathbf{Y}$ from a category $\mathbf{X}$ to $\mathbf{Y}$ is *representable* if there exists a natural isomorphism $\Pi_{O_m}: \mathrm{Hom}(O_m, O_r) \rightarrow F(O_m)$, from $\mathrm{Hom}(O_m, O_r)$ to $F(O_m)$, for objects $O_m \in \mathbf{X}$. That is, for every $y_k \in F(O_m)$ there exists a unique map $f_{r_k} \in \mathrm{Hom}(O_m, O_r)$ such that $\Pi_{O_m}(f_{r_k}) = y_k$.

Each natural transformation Π_{O_m} is with respect to O_m for a fixed O_r. For a fixed O_r, the set of natural transformations from Hom-functor to F is

$$\mathrm{Nat}(\psi_{O_r}, F) = \{ \Pi_{O_m} \mid \Pi_{O_m}: \mathrm{Hom}(O_m, O_r) \rightarrow F(O_m) \text{ for } O_m \in X\};$$

in detail, O_m "runs", for $m = 1,2,3, \dots p$, while O_r is fixed.

The Yoneda lemma guarantees that the transformation Π_{O_m} in Definition 4 is uniquely determined by an element $y_0 \in F(O_r)$.

Yoneda Lemma (Mac Lane, 1998b, p. 156)**.** Let F be an arbitrary functor from a category $\mathbf{X}$ to the category of sets, $\mathbf{SET}$, Denoted $F: \mathbf{X} \rightarrow \mathbf{SET}$. For any fixed object $O_r \in \mathbf{X}$ the set of natural transformations $\mathrm{Nat}(\psi_{O_r}, F)$ is isomorphic to the object $F(O_r)$, written

$$\mathrm{Nat}(\psi_{O_r}, F) \cong F(O_r).$$

Perception Representability

In our Scenario 1, the pair $(O_1, q_{1_0}) \equiv (\text{car}, \text{quality rep})$ is the universal element (representing object) for perception (functor) $P: \mathbf{E} \rightarrow \mathbf{C}$. For the fixed object $O_1 \equiv \text{car}$ in $\mathbf{E}$,The direct application of the Yoneda lemma tells us that there is a natural isomorphism

$\Pi_{O_m}: \mathrm{Hom}(O_m, O_1) \rightarrow P(O_m)$, from $\mathrm{Hom}(O_m, O_1)$ to $P(O_m)$, for objects $O_m \in \mathbf{C}$,

that is uniquely determined by the element $q_{1_0} \in P(O_1)$. Hence, perception P is "mathematically" representable. This mathematically demonstrates that, in general, our perception properly represents (presents) our environment to us in the form of mental representations that preserve the environmental structure (objects and their relations).

Some Consequences of Yoneda's lemma for our "Representable" Perception Functor

Theorem 1. Representable Functors preserve categorical limits.
Perception preserves the properties of universal objects, which are categorical limits.

Theorem 1.1. If a functor does not preserve limits, then it is not representable.

In our case, if our perception does not preserve universal objects for some cases, then our perception does not properly represent the "actual" situation (environment) for those cases.

Theorem 2. A functor $F: \mathbf{X} \rightarrow \mathbf{Y}$ embeds category $\mathbf{X}$ into category $\mathbf{Y}$ if it is representable.

Perception $P: \mathbf{E} \rightarrow \mathbf{C}$ is representable and therefore embeds the physical category $\mathbf{E}$ into the mental category $\mathbf{C}$.

Theorem 3. A functor $F: \mathbf{X} \rightarrow \mathbf{Y}$ is representable if and only if it has a left adjoint $G: \mathbf{Y} \rightarrow \mathbf{X}$.

Further, a corollary due to Lawvere states that any functor $F: \mathbf{X} \rightarrow \boldsymbol{Y}$ has at most one adjoint functor $G: \mathbf{Y} \rightarrow \boldsymbol{X}$, up to equivalence (Lawvere, 1963, p. 1). An adjunction is an inverse relationship between two functors.

Since $P: \mathbf{E} \rightarrow \boldsymbol{C}$ is representable, there exists a functor $B: \mathbf{C} \rightarrow \boldsymbol{E}$ that is right adjoint to P. In the following section we will explore just what this right adjoint could be, in the context of behavioral psychology.

1.3 Behavior and Categorical Adjunction

One beautiful ability, typical of our behavioral psychology, is that if we have a perceived relation $\mathrm{Pg}: P(O_n) \rightarrow P(O_m)$ in $\boldsymbol{C}$, between two objects O_n and O_m in $\mathbf{E}$, then we can behave (do physical actions) in $\mathbf{E}$ that implements the physical functional features of both O_n and O_m , and behavior that is also commensurate with how O_n and O_m are related. Could this behavior be the adjoint $B: \mathbf{C} \rightarrow \mathbf{E}$ to our perception $P: \mathbf{E} \rightarrow \mathbf{C}$, that maps perceived aspects of objects and their relationships in $\mathbf{C}$ to behavior in $\mathbf{E}$? That is, behavior that is commensurate with achieving a desired objective or goal in $\mathbf{E}$.

Definition 5 (Behavior). Given $\mathrm{P}: \mathbf{E} \rightarrow \mathbf{C}$ and any object $\mathrm{O_i} \in \mathbf{E}$ such that $\mathrm{P(O_i)} \in \mathbf{C}$ define behavior $B: \mathbf{C} \rightarrow \mathbf{E}$, a functor from a cognitive mental subspace $\mathbf{C}$ to a physical environment $\mathbf{E}$, to be the set

$$\mathrm{BP(O_i)} = \{\text{physical behaviors that are implementations of physical features of } \mathrm{O_i}\ \}.$$

Also, if $\mathrm{Pg}: \mathrm{P(O_i)} \rightarrow \mathrm{P(O_m)}$ is a relation in $\mathbf{C}$, then define

$$\mathrm{BPg} = \{\text{physical behaviorioral implementations of the relation } \mathrm{g}: \mathrm{O_i} \rightarrow \mathrm{O_m}\}.$$

In this definition we also require that $\mathrm{O_i} \subset \mathrm{BP(O_i)}$ and each behavior $\mathrm{b_{i_k}} \in \mathrm{BP(O_i)} - \mathrm{O_i}$ is an implementation of some physical feature $\mathrm{a_{i_k}} \in \mathrm{O_i}$.

We stress here that since physical behavior in $\mathrm{BP(O_i)}$ actually depends on physical use of physical features of $\mathrm{O_i}$, we must have that $\mathrm{O_i} \subset \mathrm{BP(O_i)}$. There for the canonical injection map $\theta_\mathrm{i}: \mathrm{O_i} \rightarrow \mathrm{BP(O_i)}$ always exists in $\mathbf{E}$.

We investigate the functor $\mathrm{B}: \mathbf{C} \rightarrow \mathbf{E}$ with respect to the following theorem, an augmentation of a result due to Lawvere (1963, p. 1).

Theorem 4. The functors $\mathrm{P}: \mathbf{E} \rightarrow \mathbf{C}$ and $\mathrm{B}: \mathbf{C} \rightarrow \mathbf{E}$ are adjoint if and only if for every object $\mathrm{O_i} \in \mathbf{E}$ there exists an object $\mathrm{P(O_i)} \in \mathbf{C}$ and a map $\theta_\mathrm{i}: \mathrm{O_i} \rightarrow \mathrm{BP(O_i)}$ in $\mathbf{E}$, such that for any other $\mathrm{P(O_m)} \in \mathbf{C}$ if $\mathrm{f}: \mathrm{O_i} \rightarrow \mathrm{BP(O_m)}$, then there exists a unique map $\mathrm{u}: \mathrm{P(O_i)} \rightarrow \mathrm{P(O_m)}$ in $\mathbf{C}$ such that $\mathrm{f} = \mathrm{Bu} \circ \theta_\mathrm{i}$.

Here, the map $\theta_\mathrm{i}: \mathrm{O_i} \rightarrow \mathrm{BP(O_i)}$ is universal from the object $\mathrm{O_i}$ to the functor B, for every $\mathrm{O_i} \in \mathbf{E}$ and $\mathrm{P(O_i)} \in \mathbf{C}$, and actually determines the adjunction relation between P and B. In Scenario 1, given any object $\mathrm{O_i} \in \mathbf{E}$, if a person behaves or acts with respect to $\mathrm{O_i}$ in the environment, defined by Definition 5, we are guaranteed that the canonical injection map $\theta_\mathrm{i}: \mathrm{O_i} \rightarrow \mathrm{BP(O_i)}$ satisfies the criteria of Theorem 4, so that our behavior $\mathrm{B}: \mathbf{C} \rightarrow \mathbf{E}$ is adjoint to our perception $\mathrm{P}: \mathbf{E} \rightarrow \mathbf{C}$.

The diagram in Figure 2 describes an empirical case of Theorem 4, in terms of Scenario 1, where the "behavioral" objects $\mathrm{BP(O_i)}$ and arrows $\mathrm{BPg}: \mathrm{P(O_i)} \rightarrow \mathrm{P(O_m)}$ naturally form a subcategory in category $\mathbf{E}$. If the object $\mathrm{O_4} \in \mathbf{E}$ is person A's company (revenue source), then physical features of $\mathrm{O_4}$ can be the "company facility", "personnel", "equipment", "office supplies", etc. A's perception produces some cognitive knowledge space $\mathrm{P(O_4)} \in \mathbf{C}$ of perceivable aspects of $\mathrm{O_4}$ (Falmagne & Doignon, 2011, p. 23). A's behavior $\mathrm{BP(O_4)} \in \mathbf{E}$ will consist of implementations of physical features of $\mathrm{O_4}$ to produce revenue; for instance, physical use of equipment, verbal or written communications to and from personnel, physical use of office supplies, etc. Since person A is acting in and on physical features of $\mathrm{O_4}$, we have $\mathrm{O_4} \subset \mathrm{BP(O_4)}$, so that the injection map $\theta_4: \mathrm{O_4} \rightarrow \mathrm{BP(O_4)}$ exists in $\mathbf{E}$. A's knowledge space, $\mathrm{P(O_8)} \in \mathbf{C}$, of their creditors exists, consisting of knowledge about payment options, payment schedules, payment amounts and credit reporting; and there is a map $\mathrm{d}: \mathrm{O}_4 \rightarrow \mathrm{BP}(\mathrm{O}_8)$ in $\mathbf{E}$, that exists by composition of the existing maps - canonical $\theta_8: \mathrm{O}_8 \rightarrow \mathrm{BP}(\mathrm{O}_8)$ with $s: \mathrm{O}_4 \rightarrow \mathrm{O}_8$, written $\mathrm{d} = \theta_8 \circ s$. the set $\mathrm{BP}(\mathrm{O}_8) - \mathrm{O}_8$ is the set of physical implementations of the features of the creditors, taken by A, that result in the creditors reporting the credit status of A or A's company O_4, to credit bureaus. Finally, by virtue of the map $s: \mathrm{O}_4 \rightarrow \mathrm{O}_8$, there exists the unique map, knowledge of the relation between O_4 and O_8 given by $\mathrm{Ps}: \mathrm{P}(\mathrm{O}_4) \rightarrow \mathrm{P}(\mathrm{O}_8)$ in $\mathbf{C}$ such that $\mathrm{d} = \mathrm{BPs} \circ \theta_4$, where $\mathrm{BPs}: \mathrm{BP}(\mathrm{O}_4) \rightarrow \mathrm{BP}(\mathrm{O}_8)$.

The map BPs deserves a comment here; it is A's behavior that is "commensurate" with the known relation between the company O_4 and creditors O_8. We typically act or behave in accord with the relationship between entities, in order to accomplish an objective that depends on the entities; or, to just behave "properly" in the space of possible relational behaviors $\mathrm{BP}(\mathrm{O}_4) \times \mathrm{BP}(\mathrm{O}_8)$. In this case, BPs is that "proper" behavior that will solicit good credit status reporting from creditors O_8. Mathematically, BPs is in the cross space of $\mathrm{BP}(\mathrm{O}_4)$ and $\mathrm{BP}(\mathrm{O}_8)$, written $\mathrm{BPs} \subset \mathrm{BP}(\mathrm{O}_4) \times \mathrm{BP}(\mathrm{O}_8)$, and is also a well-defined structure, if we study behavior with respect to Lawvere's Comma Category definition of the adjoint, that defines the adjoint in terms of relations (maps) instead of objects (Lawvere, 1963, p. 1).

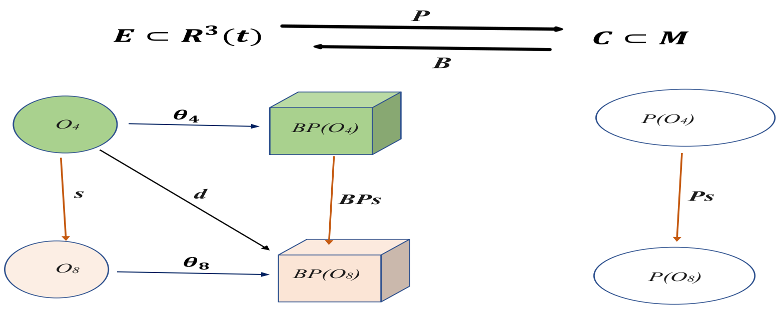

Figure 2. *This diagram illustrates the behavioral adjoint $B: \boldsymbol{C} \rightarrow \boldsymbol{E}$ to the perception functor $P: \boldsymbol{E} \rightarrow \boldsymbol{C}$. Entities $O_4, O_8 \in \boldsymbol{E}$ are person A's company and creditors, respectively and A's associated knowledge of them $P(O_4), P(O_8) \in \boldsymbol{C}$ defined as in Figure 1. The entities $BP(O_4), BP(O_8) \in \boldsymbol{E}$ are A's physical behaviors that are implementations of the physical features of $O_4, \subset BP(O_4)$ and $O_8 \subset BP(O_8)$, respectively. The injection map $\theta_4: O_4 \rightarrow BP(O_4)$ is universal from O_4 to $BP(O_4)$ and the map $Ps: P(O_4) \rightarrow P(O_8)$ is the unique map such that $d = BPs \circ \theta_4$. The map BPs is A's behavior that is "commensurate" with the relation between the company O_4 and creditors O_8.*

2. WHY MAP OBJECTS TO SETS OF RELATIONS?

Relations between "objects" are physically and psychologically significant. A little exploration with physical science will expose the wealth of physically significant relations between physical objects and entities. We will, here, focus on relations between "entities" and the psychological significance for mapping those entities to sets of relations between those entities and others, such as those constructed in the Yoneda lemma. In a simple statement,

"you can learn the contents of a person or entity B by studying the set of relations, $\mathrm{Hom}(B, X)$ or $\mathrm{Hom}(X, B)$, between B and other persons or entities X; or even $\mathrm{Hom}(B, B)$, the set of relations between B and itself".

Any relationship f between B and X comes with a set of requirements or responsibilities in order for that relationship to hold. The quality of the relationship is a result of the degree to which B and X "maintain and contribute to their own content" commensurate with creating and maintaining the relationship f. It is possible that the relationship f can be maintained, enhanced, diminished, or even dissolved, whether B and X are both people, communities, physical objects, or B a person and X a physical entity.

We provide two psychobehavioral studies applied to marketing, that are supported by constructions involving the Yoneda lemma.

Psychobehavioral Marketing 1

Hennig-Thurau, Gwinner, and Gremler (2002, p. 230) studied relationship structures and quality between businesses and customers. We interpret their study in terms of Yoneda constructs. Hennig-Thurau, et al., considered a set of relationships $\mathrm{Hom}(B, X_n)$ between a business B and its customers X_n , and studied how selected antecedents (elements) $\mathrm{a}_i \in \mathrm{F}(B)$ can solicit desired customer responses (elements) $\mathrm{r}_{i_\mathrm{n}} \in \mathrm{F}(X_n)$ mediated by a relationship $f_i \in \mathrm{Hom}(B, X_n)$. The particular customer responses were "repeat purchase", "trust", "satisfaction", and "X_n tells a friend Y_n about business", which can be denoted by a communication arrow $c: X_n \rightarrow Y_n$.
The Hom-functor $\mathrm{Hom}(B, \_\_)$ maps customers, from the physical category of customers and businesses, to the set of relationships between those customers and a particular business B. Further, the Hom-functor maps the communication relationship $c: X_n \rightarrow Y_n$ to the composite relation $c \circ f_i$ between the business and customer Y_n , satisfying definition 3. The functor, $\mathrm{F}: \mathbf{E} \rightarrow \mathbf{K}$, from a category of physical entities $\mathbf{E}$ (businesses and customers) to a category of antecedents and costumer responses, $\mathbf{K}$, can be defined as a "piece-wise" functor so that for any business B and customer X_n in $\mathbf{E}$,

$$\mathrm{F}(B) = \{\text{product reliability, information cofidentiality, customer perks}\},$$

and

$$\mathrm{F}(X_n) = \{\text{repeat purchases, trust, satisfaction, word of mouth referal}\}.$$

In the isomorphic structure, $\mathrm{Nat}(\mathrm{Hom}(\mathrm{B}\,, \mathrm{X_n}),\ \mathrm{F}(\mathrm{X_n})) \cong \mathrm{F}(\mathrm{B})$, is the description of the empirical case for applying antecedents in $\mathrm{F}(\mathrm{B})$ to solicit customer responses from $\mathrm{F}(\mathrm{X_n})$ that are commensurate with the business-customer relations in $\mathrm{Hom}(\mathrm{B}\,, \mathrm{X_n})$.

In the case of Hennig-Thurau, Gwinner, and Gremler, any particular business or customer can be seen as a universal element.

Psychobehavioral Marketing 2

The self-concept and its associations is another psychological context where Yoneda lemma constructs are inherent. Rosenberg states that the self-concept is the totality of the individual's thought and feelings having reference to himself as an object (Rosenberg, 1979, p 9). The self-concept is how we define ourselves through identities (meaning) that define our individual characteristics, our roles, our social categories, etc (Stets & Burke, 2014, p. 409). The self-concept (self-image) requires a set of reflexive processes on how a person X_n views and relates to their self, denoted by the set $\mathrm{Hom}(X_n, X_n)$. In a marketing study, Upamannyu, Mathur, and Bhakar (2014, p. 308) studied the relation between the self-concept (actual self-image and ideal self-image) and brand-image. The concept of brand functionality or utility was not the focus of their study, only brand-image. The concept of congruence was defined between a consumer's self-concept and the consumer's perception of a brand (brand-image), and found that this "congruence" determines positive attitudes, preferences, and behaviors towards that brand. Here, for any customer X_n the self-concept can be the set of self-images defined by

$$\mathrm{Hom}(X_n, X_n) = \{\mathrm{v_j}: X_n \rightarrow X_n \big| \, X_n \text{ defines (veiws) them self as } \mathrm{v_j}\},$$

where $\mathrm{v_j}$ can be a personal characteristic, professional or social title (role), nationality, ethnicity, gender; for example, X_n defines themselves as intelligent, sophisticated, a business person, Nigerian, Hausa, etc.

For any brand Z, the brand-image can be a set of intangible properties of Z. The brand Z can be Coca Cola, Apple, Mercedes Benz, etc. The set of intangible properties of Z can be "Z is professional", "Z is prestigious", "Z is All American" , etc (Upamannyu, Mathur, & Bhakar, 2014, p. 308). The set of intangibles is defined by the person $X_n's$ perception of that brand, and can be denoted by the perception functor

$$\mathrm{P(Z)} = \{\mathrm{a_j} \mid \mathrm{a_j} \text{ intangible property of Z}\},$$

where $\mathrm{P}: \mathbf{E} \rightarrow \mathbf{C}$ is a functor from a physical (environmental) category of brands and customers, $\mathbf{E}$, into $X_n's$ categorical cognitive space $\mathbf{C}$. Intangibles $\mathrm{a_j}$ can be "All American", "prestigious", "professional", etc. We can denote and define the congruence relation by

$\mathrm{Hom}(X_n, X_n) \sim \mathrm{P(Z)}$ if and only if X_n believes that the brand-image is very similar to their own self-image.

According to statistical results in (Upamannyu, Mathur, & Bhakar, 2014, p. 308), there is a positive relation between congruence and customer positive attitudes, preference, and behaviors towards brand purchases. Here, we can generalize the study of the relation between congruence and tangible customer behavior denoted by the possible existence of an isomorphism

$$\Psi: \mathrm{Cong}(\mathrm{Hom}(X_n, X_n), \mathrm{P(Z)}) \cong \mathrm{F(Z)},$$

where the object $\mathrm{Cong}(\mathrm{Hom}(X_n, X_n), \mathrm{P(Z)})$ is the set of congruences between $X_n's$ self-image and $X_n's$ perception of brand Z.

Since we have a behavior functor B that is adjoint to P, define a customer behavior functor $\mathrm{F} = \mathrm{BP}$, $\mathrm{F}: \mathbf{E} \to \mathbf{S}$, from the physical category of brands into a physical subcategory $\mathbf{S}$ of environment $\mathbf{E}$, denoted $\mathbf{S} \subset \mathbf{E}$. The expression of $X_n's$ customer behaviors is of a physical nature, therefore we define the objects of $\mathbf{S}$ to be sets of customer behaviors; for example the object $\mathrm{F(Z)}$ in $\mathbf{S}$ can be ,

$$\mathrm{F(Z)} = \{\text{expressed attitudes, preference, repeat purchases, and customer referrals}\}.$$

The objects $\mathrm{P(Z)}$, $\mathrm{Hom}(X_n, X_n)$, and $\mathrm{Cong}(\mathrm{Hom}(X_n, X_n), \mathrm{P(Z)})$ are all $X_n's$ views, self-perceptions (conceptions), and beliefs that are of a mental nature, and therefore are mental objects that “reside” in the cognitive space (category) $\mathbf{C}$ of customer X_n. Now, the isomorphism Ψ has more meaning. It can be defined as a generalized behavior-map, mapping congruences in “$\mathbf{Cong}$”, from the mental space into the physical environment.

The “Yoneda” constructs support the all-important and integral part of any marketing strategy for any business that involves accessing or “shaping” the customer’s definitions and perceptions of themselves $\mathrm{Hom}(X_n, X_n)$, creating brand perceptions $\mathrm{P}(Z)$ that “match” customer self-perceptions, and using that match (congruence) $\mathrm{Cong}(\mathrm{Hom}(X_n, X_n), \mathrm{P}(Z))$ to solicit prescribed desired customer behavioral responses $F(Z)$ towards the respective business brand Z. Here, the business is interested in an isomorphic map (stimulus), such as Ψ , that would unambiguously and uniquely map each perceived customer congruence $\mathrm{c}_j \in \mathrm{Cong}(\mathrm{Hom}(X_n, X_n), \mathrm{P}(Z))$ to a customer behavior $x_j \in F(Z)$.

DISCUSSION

We investigated the presence of some mathematical structures inherent in the anatomy of our cognitive and behavioral psychology, that we intentionally or uncounsciously employ. Mathematically, a “universal object” is said to be an optimal solution to a given problem, and the adjoint relationship between Functors arises from the construction of these universal objects (Phillips & Wilson, 2011) . Whenever an individual focuses on an object (the focus), or an objective, and is also cognizant of other objects that are “uniquely” related to the focus, that focus becomes a universal object, both in the cognitive mind and behavioral space of the person postulating such a construct. Subsequent to perception and mental construction of this “optimal solution” to a given problem, a set of behaviors can be employed, unique to solving the problem or achieving the objective.

While the adjoint behavioral functor exists, there is another psychological class of “objects” to be considered that can impede, preclude, or promote behavior. Emotional content is that class. For instance, many students are faced with the objective of passing a math course or courses in order to graduate. For the student, the math course and passing becomes the universal object, and usually present are all supportive entities uniquely related to doing well in the course. If in the student’s mind, the very idea of the course is shrouded in anxiety and fear, that emotional charge can impede or preclude any behavior commensurate with passing the course. In the political arena, politicians may be cognizant that a combined (glued) bipartisan solution is the optimal solution to a problem, but fear of back-lash from their constituents can impede or preclude political behavior that implements the bipartisan solution. Feelings of joy, confidence, or courage associated with universal objects can motivate behavioral implementation of optimal solutions, and can promote persistence of behaviors towards achieving objectives.

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